Global convergence of a reaction–diffusion predator–prey model with stage structure for the predator

Rui Xu a,*,1, M.A.J. Chaplain b, F.A. Davidson b

a Department of Mathematics, Institute of Shijiazhuang Mechanical Engineering, 97 Heping West Road, Shijiazhuang 050003, Hebei Province, PR China
b Department of Mathematics, University of Dundee, Dundee DD1 4HN, UK

Abstract

A Lotka–Volterra type reaction–diffusion predator–prey model with stage structure for the predator and time delay due to maturity is investigated. By successively modifying the coupled lower–upper solution pairs, sufficient conditions independent of the effect of spatial diffusion are derived for the global convergence of positive solutions to the proposed problem.

© 2005 Elsevier Inc. All rights reserved.

Keywords: Reaction–diffusion; Stage structure; Time delay

1. Introduction

Stage-structured models have received great attention in recent years (see, for example, [1–11,16–22]). The pioneering work of Aiello and Freedman [1] on a single species growth model with stage structure represents a mathematically more careful and biologically meaningful formulation approach. In [1], a model of single species population growth incorporating stage structure as a reasonable generalization of the classical logistic model was formulated and discussed. This model assumes an average age to maturity which appears as a constant time delay reflecting a delayed birth of immatures and a reduced survival of immatures to their maturity. The model takes the form

\[
\begin{align*}
\dot{u}_i(t) &= axm(t) - \gamma u_i(t) - x e^{-\tau} u_m(t - \tau), \\
\dot{u}_m(t) &= x e^{-\tau} u_m(t - \tau) - \beta u^2_m(t), \quad t > \tau,
\end{align*}
\]

where \(u_i(t)\) denotes the immature population density, \(u_m(t)\) represents the mature population density, \(a > 0\) represents the birth rate, \(\gamma > 0\) is the immature death rate, \(\beta > 0\) is the mature death and overcrowding rate,

* Corresponding author.
E-mail address: rxu88@yahoo.com.cn (R. Xu).

1 The first author’s work was supported by the National Natural Science Foundation of China (No. 10471066).
\( \tau \) is the time to maturity. The term \( x e^{-\tau} u_m(t - \tau) \) represents the immatures who were born at time \( t - \tau \) and survive at time \( t \) (with the immature death rate \( \gamma \)), and therefore represents the transformation of immatures to matures. Aiello and Freedman in model (1.1) assume that the maturation delay \( \tau \) is know exactly and that all individuals take this amount of time to mature. In [3], Al-Omari and Gourley studied a more general model than model (1.1) by replacing the term \( x e^{-\tau} u_m(t - \tau) \) with a distribution of maturation times, weighted by a probability density function. They assumed that at time \( t \) the number that become mature, per unit time, is

\[
\int_0^\infty x u_m(t - s) f(s) e^{-\tau s} \, ds.
\]

The term \( x u_m(t - s) \) is the number born at time \( t - s \) per unit time, and is taken as proportional to the number of mature adults then around. The function \( f(s) \) is the probability of taking time \( s \) to mature, and \( e^{-\tau s} \) is the probability of an individual born at time \( t - s \) still being alive at time \( t \). Individuals becoming mature at time \( t \) could have been born at any time prior to this, and the integral totals up the contributions from all previous times. Therefore, the model (1.1) is generalized to

\[
\begin{align*}
\dot{u}_1(t) &= xu_m(t) - \gamma u_1(t) - \alpha \int_0^\infty u_m(t - s) f(s) e^{-\tau s} \, ds, \\
\dot{u}_m(t) &= \alpha \int_0^\infty u_m(t - s) f(s) e^{-\tau s} \, ds - \beta u_m^2(t),
\end{align*}
\]

where \( \int_0^\infty f(s) \, ds = 1 \) and \( f(s) \geq 0 \).

The effect of spatial dispersion on population dynamics has received considerable recent attention. In this situation, the governing equations for the population densities are described by a system of reaction–diffusion equations. An ecological interesting and mathematically challenging problem is to determine whether the time-dependent solution converges to a positive steady state solution, and to which one, if these are multiple, for a given class of initial data (see, for example, [12–14]).

Motivated by the work on stage-structured competition model by Al-Omari and Gourley [3] and the work on predator–prey model without stage structure by Pao [12–14], in the present paper, we discuss the following delayed reaction–diffusion Lotka–Volterra type model for prey and adult predator interaction:

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= D_1 \Delta u_1 + u_1(t, x) (r_1 - a_{11} u_1(t, x) - a_{12} u_2(t, x)), \quad t > 0, \ x \in \Omega, \\
\frac{\partial u_2}{\partial t} &= D_2 \Delta u_2 + u_2(t, x) \int_0^\infty f(s) e^{-\tau s} u_1(t - s, x) u_2(t - s, x) \, ds \\
&\quad - r_2 u_2(t, x) - a_{22} u_2^2(t, x), \quad t > 0, \ x \in \Omega, \\
\frac{\partial u_1}{\partial v} &= \frac{\partial u_2}{\partial v} = 0, \quad t > 0, \ x \in \partial \Omega, \\
u_1(t, x) &= \phi_1(t, x), u_2(t, x) = \phi_2(t, x), \quad t \leq 0, \ x \in \Omega.
\end{align*}
\]

In problem (1.3), \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \), where \( \partial / \partial v \) denotes the outward normal derivative on \( \partial \Omega \). The boundary conditions in (1.3) imply that the populations do not move across the boundary \( \partial \Omega \). The parameters \( r_1, r_2, a_{11}, a_{12}, a_{22}, \alpha \) and \( \gamma \) are positive constants. \( u_1(t, x) \) represents the density of the prey population at time \( t \) and location \( x \), \( u_2(t, x) \) denotes the density of the mature predator population at time \( t \) and location \( x \), respectively. The data \( \phi_i(t, x) \) \( (i = 1, 2) \) are nonnegative and Hölder continuous and satisfy \( \partial \phi_i / \partial v = 0 \) in \( (-\infty, 0) \times \partial \Omega \). The model is derived under the following assumptions.

(A1) The prey population: the growth of the species is of Lotka–Volterra nature. The parameters \( r_1, a_{11} \) and \( D_1 \) are the intrinsic growth rate, intra-specific competition rate and diffusion rate, respectively.

(A2) The predator population: \( a_{12}, a_{12}/a_{12}, r_2, \) and \( a_{22} \) are the capturing rate, conversion rate, death rate and intra-specific competition rate of the mature predator, respectively; \( \gamma > 0 \) is the death rate of the immature predator population, \( D_2 \) is the diffusion rate of the mature population. The term \( Xu_1(t - s, x)u_2(t - s, x) \) is the number born at time \( t - s \) and location \( x \) per unit time, and is taken as proportional to the number of the prey and mature predator then around. \( f(s)ds \) denotes the probability that the
maturation time is between \( s \) and \( s + ds \) with \( ds \) infinitesimal, and \( \int_0^\infty f(s)\, ds = 1 \). \( e^{-\nu t} \) is the probability of an individual born at time \( t - s \) still being alive at time \( t \). Individuals becoming mature at time \( t \) could have been born at any time prior to this, and the integral totals up the contributions from all previous times.

Following [3], in this paper, for technical reasons we always assume that the kernel \( f(s) \) has compact support, that is, \( f(s) = 0 \) for all \( s \geq \tau \), for some \( \tau > 0 \), and normalized such that \( \int_0^\infty f(s)\, ds = 1 \). This is also biologically reasonable. In this case, problem (1.3) becomes

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= D_1 \Delta u_1 + u_1(t,x)(r_1 - a_{11}u_1(t,x) - a_{12}u_2(t,x)), \\
\frac{\partial u_2}{\partial t} &= D_2 \Delta u_2 + \alpha \int_0^t f(s)e^{-\nu s}u_1(t-s,x)u_2(t-s,x)\, ds - r_2u_2(t,x) - a_{22}u_2^2(t,x)
\end{align*}
\]  

(1.4)

for \( t > 0 \) and \( x \in \Omega \), with homogeneous Neumann boundary conditions

\[
\frac{\partial u_1}{\partial v} = \frac{\partial u_2}{\partial v} = 0, \quad t > 0, \ x \in \partial \Omega,
\]  

(1.5)

and initial conditions

\[
u_1(t,x) = \phi_1(t,x), \quad u_2(t,x) = \phi_2(t,x), \quad t \in [-\tau,0], \ x \in \bar{\Omega},
\]  

(1.6)

where \( \phi_i \) (\( i = 1, 2 \)) are nonnegative and Hölder continuous with \( \phi_i(0,x) \neq 0 \).

In this paper, for system (1.4) we always assume that the following assumption holds:

\[(H1) \ f(t) \text{ is piecewise continuous in } [0, \tau] \text{ and has the property: } f(t) \geq 0, \int_0^\infty f(t)\, dt = 1.\]

This paper is organized as follows. In the next section, we discuss the existence, uniqueness, positivity and boundedness of solutions of problem (1.4)–(1.6). By successively modifying the coupled lower–upper solution pairs, sufficient conditions independent of the effect of spatial diffusion are derived for the global convergence of the positive solutions of problem (1.4)–(1.6). A brief discussion is presented in Section 3.

2. Global convergence

In this section, we first discuss the existence, uniqueness, positivity and boundedness of problem (1.4)–(1.6). To do so, we need the following concepts and results.

**Definition 2.1.** A pair of functions \( \bar{\nu} = (\bar{\nu}_1, \bar{\nu}_2), \tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2) \) in \( C([0, \infty) \times \bar{\Omega}) \cap C^{1,2}((0, \infty) \times \Omega) \) are called coupled upper and lower solutions of system (1.4)–(1.6) if \( \bar{\nu}_i \leq \tilde{\nu}_i (i = 1, 2) \) in \( [-\tau, \infty) \times \bar{\Omega} \) and if for all \( \bar{\nu}_i \leq \psi_i \leq \tilde{\nu}_i \) (\( i = 1, 2 \)), the following differential inequalities hold:

\[
\begin{align*}
\frac{\partial \bar{\nu}_1}{\partial t} &\geq D_1 \Delta \bar{\nu}_1 + \bar{u}_1(t,x)(r_1 - a_{11}\bar{u}_1(t,x) - a_{12}\bar{u}_2(t,x)), \\
\frac{\partial \bar{\nu}_2}{\partial t} &\geq D_2 \Delta \bar{\nu}_2 + \alpha \int_0^t f(s)e^{-\nu s}\psi_1\psi_2\, ds - r_2\bar{u}_2 - a_{22}\bar{u}_2^2, \\
\frac{\partial \tilde{\nu}_1}{\partial t} &\leq D_1 \Delta \tilde{\nu}_1 + \tilde{u}_1(t,x)(r_1 - a_{11}\tilde{u}_1(t,x) - a_{12}\tilde{u}_2(t,x)), \\
\frac{\partial \tilde{\nu}_2}{\partial t} &\leq D_2 \Delta \tilde{\nu}_2 + \alpha \int_0^t f(s)e^{-\nu s}\psi_1\psi_2\, ds - r_2\tilde{u}_2 - a_{22}\tilde{u}_2^2
\end{align*}
\]

(2.1)

for \( (t, x) \in (0, \infty) \times \Omega \), and

\[
\begin{align*}
\frac{\partial \bar{\nu}_1}{\partial v} &\leq 0 \leq \frac{\partial \tilde{\nu}_1}{\partial v} \quad (i = 1, 2), \ (t, x) \in (0, \infty) \times \partial \Omega, \\
\bar{u}_i(t,x) &\leq \phi_i(t,x) \leq \tilde{u}_i(t,x) \quad (i = 1, 2), \ (t, x) \in [-\tau, 0] \times \bar{\Omega}.
\end{align*}
\]

(2.2)
The following lemma then follows from Theorem 3.4 in [15].

**Lemma 2.1.** Let \( \bar{u} \) and \( \hat{u} \) be a pair of coupled upper and lower solutions for problem (1.4)–(1.6) and suppose that the initial functions \( \phi_i \) \((i = 1, 2)\) are Hölder continuous in \([-\tau, 0] \times \bar{\Omega}\). Then problem (1.4)–(1.6) has exactly one regular solution \( u(t, x) = (u_1(t, x), u_2(t, x)) \) satisfying \( \bar{u} \leq u \leq \hat{u} \) in \([-\tau, \infty) \times \bar{\Omega}\).

It is readily seen that \( 0 = (0, 0) \) and \( \mathbf{K} = (K_1, K_2) \) are a pair of coupled lower–upper solutions of problem (1.4)–(1.6), where

\[
K_1 = \max \left\{ \frac{r_1}{a_{11}}, \sup_{-\tau \leq \theta \leq 0} \| \phi_1(\theta, \cdot) \|_{C(\bar{\Omega}, R)} \right\},
\]

\[
K_2 = \max \left\{ \alpha K_1 \int_0^\tau f(s)e^{-\gamma s}\, ds / a_{22}, \sup_{-\tau \leq \theta \leq 0} \| \phi_2(\theta, \cdot) \|_{C(\bar{\Omega}, R)} \right\}.
\]

Hence we have \( 0 \leq u(t, x) \leq K_i \) for \((t, x) \in [-\tau, \infty) \times \bar{\Omega}\). Also, by the maximum principle, if \( \phi_i(0, x) \neq 0 \) \((i = 1, 2)\), we have \( u_i(t, x) > 0 \) \((i = 1, 2)\) for all \( t > 0, x \in \bar{\Omega}\).

In order to discuss the global convergence of positive solutions of problem (1.4)–(1.6), we first prove a result for a scalar reaction–diffusion equation with homogeneous Neumann boundary conditions.

**Lemma 2.2.** Let the initial function \( \phi \) be Hölder continuous in \([-\tau, 0] \times \bar{\Omega}\). Assume that \( A_1 \geq 0, B > 0, A_2 > 0, \) and \( f(s) \) is defined as in (H1). Let \( u(t, x) \) be a nonnegative nontrivial solution of the following scalar problem:

\[
\frac{\partial u}{\partial t} = D\Delta u + B \int_0^\tau f(s)e^{-\gamma s}u(t-s, x)\, ds - A_1 u(t, x) - A_2 u^2(t, x), \quad (t, x) \in (0, \infty) \times \bar{\Omega},
\]

\[
\frac{\partial u}{\partial v} = 0, \quad (t, x) \in (0, \infty) \times \partial \Omega,
\]

\[
u(t, x) = \phi(t, x) \geq 0, \quad \phi(0, x) \neq 0, (t, x) \in [-\tau, 0] \times \bar{\Omega}.
\]

We have

(i) if \( B \int_0^\tau f(s)e^{-\gamma s}\, ds > A_1 \), then \( \lim_{t \to +\infty} u(t, x) = (B \int_0^\tau f(s)e^{-\gamma s}\, ds - A_1) / A_2 \) uniformly for \( x \in \Omega\);

(ii) if \( B \int_0^\tau f(s)e^{-\gamma s}\, ds < A_1 \), then \( \lim_{t \to +\infty} u(t, x) = 0 \) uniformly for \( x \in \Omega\).

**Proof.** By the strong-maximum principle, we see that \( u(t, x) \) is positive in \((0, +\infty) \times \Omega\) if \( \phi(0, x) \neq 0 \). Therefore, there exist \( \delta > 0, t_0 > 0 \) such that \( u(t, x) \geq \delta, (t, x) \in [t_0, t_0 + \tau] \times \bar{\Omega}\). We are now concerned with (2.4) in the domain \([t_0 + \tau, +\infty) \times \bar{\Omega}\). Suppose that \( B \int_0^\tau f(s)e^{-\gamma s}\, ds > A_1 \). Let

\[
M = \max \left\{ \max_{[t_0, t_0 + \tau] \times \bar{\Omega}} u(t, x), \left( B \int_0^\tau f(s)e^{-\gamma s}\, ds - A_1 \right) / A_2 \right\}.
\]

Let \( \epsilon > 0 \) satisfy \( \epsilon < \delta \) and \( \epsilon < (B \int_0^\tau f(s)e^{-\gamma s}\, ds - A_1) / A_2 \). Take \( \tilde{\epsilon} = \epsilon, \tilde{\tilde{\epsilon}} = M \). Thus, it is easy to verify that

\[
B \epsilon \int_0^\tau f(s)e^{-\gamma s}\, ds - A_1 \epsilon - A_2 \epsilon^2 \geq 0,
\]

\[
BM \int_0^\tau f(s)e^{-\gamma s}\, ds - A_1 M - A_2 M^2 \leq 0.
\]

Therefore, \( \tilde{\epsilon} \) and \( \tilde{\tilde{\epsilon}} \) are a pair of coupled upper and lower solutions of Eq. (2.4) in \([t_0 + \tau, +\infty) \times \bar{\Omega}\). By similar arguments to those in the proof of Theorem 3.3 in [12], we see that Eq. (2.4) has quasisolutions \( \tilde{\epsilon} \) and \( \epsilon \) (see [12] for a definition of these) such that \( 0 < \epsilon \leq \tilde{\epsilon} \leq \tilde{\tilde{\epsilon}} \leq M \) and

\[
B \epsilon \int_0^\tau f(s)e^{-\gamma s}\, ds - A_1 \epsilon - A_2 \epsilon^2 = 0,
\]

\[
B \epsilon \int_0^\tau f(s)e^{-\gamma s}\, ds - A_1 \epsilon - A_2 \epsilon^2 = 0.
\]
If \( B \int_0^s f(s)e^{-\gamma s} ds > A_1 \), then system (2.5) has a unique positive solution \( \bar{c} = \zeta = (B \int_0^s f(s)e^{-\gamma s} ds - A_1)/A_2 \). Also,

\[
\lim_{t \to +\infty} u(t, x) = \frac{B \int_0^s f(s)e^{-\gamma s} ds - A_1}{A_2} \quad \text{uniformly for } x \in \Omega.
\]

If \( B \int_0^s f(s)e^{-\gamma s} ds < A_1 \). Let \( \bar{c} = L = \|\phi\|_\infty, \hat{c} = 0 \). Then \( \bar{c} \) and \( \hat{c} \) are a pair of coupled upper and lower solutions of Eq. (2.4) in \( [t_0 + \tau, +\infty) \times \Omega \). Noting that \( 0 \) is the unique nonnegative solution of equation \( c(B \int_0^s f(s)e^{-\gamma s} ds - A_1 - A_2c) = 0, c \in (0, L) \), we have

\[
\lim_{t \to +\infty} u(t, x) = 0 \quad \text{uniformly for } x \in \Omega.
\]

This completes the proof. \( \square \)

It is easy to see that system (1.4) possesses a trivial uniform equilibrium \( E_0(0, 0) \) and a semi-trivial uniform equilibrium \( E_2(r_1/a_{11}, 0) \). If the following holds:

(H2) \( r_1 \int_0^s f(s)e^{-\gamma s} ds > r_2 a_{11} \),

then (1.4) also has a unique positive uniform equilibrium \( E^*(u^*_1, u^*_2) \), where

\[
u^*_1 = \frac{r_1 a_{22} + r_2 a_{11}}{a_{11} a_{22} + \alpha a_{12} \int_0^s f(s)e^{-\gamma s} ds}, \quad u^*_2 = \frac{r_1 \int_0^s f(s)e^{-\gamma s} ds - r_2 a_{11}}{a_{11} a_{22} + \alpha a_{12} \int_0^s f(s)e^{-\gamma s} ds}.
\]

We are now in a position to state and prove our result on the global convergence of the positive solutions of problem (1.4)–(1.6). The method of proof is to use the coupled upper–lower solution technique outlined for the scalar case above.

**Theorem 2.1.** Let the initial functions \( \phi_i \) \((i = 1, 2)\) be Hölder continuous in \([-\tau, 0] \times \Omega\), with \( \phi_i(t, x) \geq 0 \), \( \phi_i(0, x) \neq 0 \). Let \((u_1(t, x), u_2(t, x))\) satisfy (1.4)–(1.6). In addition to (H1)–(H2), assume further that

(H3) \( a_{11} a_{22} > a_{12} \int_0^s f(s)e^{-\gamma s} ds \).

Then

\[
\lim_{t \to +\infty} u_1(t, x) = u^*_1, \quad \lim_{t \to +\infty} u_2(t, x) = u^*_2
\]

uniformly for \( x \in \Omega \).

**Proof.** Denote

\[
\bar{U}_i = \limsup_{t \to +\infty} \max_{x \in \Omega} u_i(t, x), \quad \underline{U}_i = \liminf_{t \to +\infty} \min_{x \in \Omega} u_i(t, x) \quad (i = 1, 2).
\]

We now claim that \( \bar{U}_1 = \underline{U}_1 = u^*_1, \bar{U}_2 = \underline{U}_2 = u^*_2 \). Let \((\bar{u}_1(t, x), \bar{u}_2(t, x))\) be the solution of the following problem:

\[
\begin{align*}
\frac{\partial \bar{u}_1(t, x)}{\partial t} &= \Delta \bar{u}_1(t, x) + \bar{u}_1(t, x)[r_1 - a_{11} \bar{u}_1(t, x)], \quad t > 0, \ x \in \Omega, \\
\frac{\partial \bar{u}_2(t, x)}{\partial t} &= \Delta \bar{u}_2(t, x) + \varepsilon \int_0^t f(s)e^{-\gamma s} \bar{u}_1(t - s, x) \bar{u}_2(t - s, x) ds \\
&\quad - r_2 \bar{u}_2(t, x) - a_{22}(\bar{u}_2(t, x))^2, \quad t > 0, \ x \in \Omega, \\
\frac{\partial \bar{u}_1(t, x)}{\partial v} &= \frac{\partial \bar{u}_2(t, x)}{\partial v} = 0, \quad t > 0, \ x \in \partial \Omega, \\
\bar{u}_1(t, x) &= K_1, \bar{u}_2(t, x) = K_2, \quad t \in [-\tau, 0], \ x \in \Omega,
\end{align*}
\]
where $K_1$, $K_2$ are defined in (2.3). Then $(0, 0)$ and $(u_1^{(1)}, u_2^{(1)})$ are a pair of coupled lower and upper solutions of problem (1.4)–(1.6). Therefore, by Lemma 2.1 we have

$$0 \leq u_1(t, x) \leq u_1^{(1)}(t, x), \quad 0 \leq u_2(t, x) \leq u_2^{(1)}(t, x), \quad (t, x) \in [-\tau, \infty) \times \Omega.$$ 

It follows from the first equation of problem (2.7) that

$$\lim_{t \to +\infty} \bar{u}_1^{(1)}(t, x) = \frac{r_1}{a_{11}} := M_1^{u_1} \quad \text{uniformly for } x \in \Omega.$$ 

Hence, for $\forall \varepsilon > 0$ sufficiently small, there exists a $t_1 > 0$ such that

$$\max_{x \in \Omega} \bar{u}_1^{(1)}(t, x) < M_1^{u_1} + \varepsilon \quad \text{for } t > t_1. \quad (2.8)$$

Since $\varepsilon > 0$ is arbitrary and sufficiently small, we can conclude that

$$\bar{U}_1 = \limsup_{t \to +\infty} \max_{x \in \Omega} u_1(t, x) \leq \frac{r_1}{a_{11}} = M_1^{u_1}. \quad (2.9)$$

We consider the following auxiliary problem:

$$\frac{\partial \omega_2^{(1)}}{\partial t} = \Delta \omega_2^{(1)} + z(M_1^{u_1} + \varepsilon) \int_0^t f(s)e^{-r_1 s} \omega_2^{(1)}(t - s, x) \, ds$$

$$- r_2 \omega_2^{(1)}(t, x) - a_{22}(\omega_2^{(1)}(t, x))^2, \quad t > t_1, \ x \in \Omega,$$

$$\frac{\partial \omega_2^{(1)}}{\partial v} = 0, \quad t > t_1, \ x \in \partial \Omega,$$

$$\omega_2^{(1)}(t, x) = K_2, \quad t \in [-\tau, t_1], \ x \in \Omega.$$ 

By Lemma 2.2 we derive from (2.10) that

$$\lim_{t \to +\infty} \omega_2^{(1)}(t, x) = \frac{z(M_1^{u_1} + \varepsilon) \int_0^t f(s)e^{-r_1 s} \, ds - r_2}{a_{22}}$$

uniformly for $x \in \Omega$. Therefore, for $\forall \varepsilon > 0$ sufficiently small, there is a $t_2 > t_1$ such that

$$\max_{x \in \Omega} \bar{u}_2^{(1)}(t, x) \leq \frac{z(M_1^{u_1} \int_0^t f(s)e^{-r_1 s} \, ds - r_2}{a_{22}} + \varepsilon \quad \text{for } t > t_2. \quad (2.11)$$

Since it is true for arbitrary $\varepsilon > 0$ sufficiently small, we obtain that

$$\bar{U}_2 = \limsup_{t \to +\infty} \max_{x \in \Omega} u_2(t, x) \leq \frac{z(M_1^{u_1} \int_0^t f(s)e^{-r_1 s} \, ds - r_2}{a_{22}} := M_1^{u_2}. \quad (2.12)$$

Let $(u_1^{(1)}(t, x), u_2^{(1)}(t, x))$ be the solution of the following problem:

$$\frac{\partial u_1^{(1)}}{\partial t} = \Delta u_1^{(1)} + u_1^{(1)}(t, x)[r_1 - a_{11}u_1^{(1)}(t, x) - a_{12}u_2^{(1)}(t, x)], \quad t > t_2, \ x \in \Omega,$$

$$\frac{\partial u_2^{(1)}}{\partial t} = \Delta u_2^{(1)} + z \int_0^t f(s)e^{-r_1 s}u_1^{(1)}(t - s, x)u_2^{(1)}(t - s, x) \, ds$$

$$- r_2 u_2^{(1)}(t, x) - a_{22}(u_2^{(1)}(t, x))^2, \quad t > t_2, \ x \in \Omega,$$

$$\frac{\partial u_2^{(1)}}{\partial v} = \frac{\partial u_2^{(1)}}{\partial v} = 0, \quad t > t_2, \ x \in \partial \Omega,$$

$$u_1^{(1)}(t, x) = \frac{1}{2} u_1(t, x), \quad u_2^{(1)}(t, x) = \frac{1}{2} u_2(t, x), \quad t \in [-\tau, t_2], \ x \in \Omega.$$
Then \((\bar{u}_1^{(1)}, \bar{u}_2^{(1)})\) and \((\bar{u}_1^{(1)}, \bar{u}_2^{(1)})\) are a pair of coupled lower and upper solutions of problem (1.4)–(1.6). By Lemma 2.1 it follows that
\[
\bar{u}_1^{(1)}(t, x) \leq u_1(t, x) \leq \bar{u}_1^{(1)}(t, x), \quad \bar{u}_2^{(1)}(t, x) \leq u_2(t, x) \leq \bar{u}_2^{(1)}(t, x).
\]

For \(\forall \varepsilon > 0\) sufficiently small, we derive from (2.11) and the first equation of system (2.13) that for \(t > t_2, x \in \Omega\),
\[
\frac{\partial u_1^{(1)}}{\partial t} \geq \Delta u_1^{(1)} + u_1^{(1)}(t, x)[r_1 - a_{11}u_1^{(1)}(t, x) - a_{12}(M_1^{\varepsilon^2} + \varepsilon)].
\]

By comparison we have \(u_1^{(1)}(t, x) \geq v_1^{(1)}(t, x), t > t_2, x \in \Omega\), where \(v_1^{(1)}(t, x)\) is the solution of the following problem:
\[
\begin{align*}
\frac{\partial v_1^{(1)}}{\partial t} &= \Delta v_1^{(1)} + v_1^{(1)}(t, x)[r_1 - a_{11}v_1^{(1)}(t, x) - a_{12}(M_1^{\varepsilon^2} + \varepsilon)], \quad t > t_2, \ x \in \Omega, \\
\frac{\partial v_1^{(1)}}{\partial v} &= 0, \quad t > t_2, \ x \in \partial \Omega, \\
v_1^{(1)}(t, x) &= \frac{1}{2} u_1(t, x), \quad (t, x) \in [-\tau, t_2] \times \bar{\Omega}.
\end{align*}
\]

It follows from (2.14) that
\[
\lim_{t \to +\infty} v_1^{(1)}(t, x) = \frac{r_1 - a_{12}(M_1^{\varepsilon^2} + \varepsilon)}{a_{11}} \text{ uniformly for } x \in \bar{\Omega}.
\]

Since it is true for arbitrary \(\varepsilon > 0\) sufficiently small, we therefore derive that
\[
\bar{U}_1 = \liminf_{t \to +\infty} \min_{x \in \Omega} u_1(t, x) \geq \frac{r_1 - a_{12}M_1^{\varepsilon^2}}{a_{11}} := N_1^{u_1},
\]
and for \(\forall \varepsilon > 0\) sufficiently small, there is a \(t_3 > t_2\) such that if \(t > t_3\),
\[
\min_{x \in \Omega} u_1^{(1)}(t, x) \geq N_1^{u_1} - \varepsilon.
\]

For \(\forall \varepsilon > 0\) sufficiently small, it follows from (2.16) and the second equation of (2.13) that for \(t > t_3 + \tau, x \in \Omega\),
\[
\frac{\partial u_2^{(1)}}{\partial t} = \Delta u_2^{(1)} + \alpha \int_0^t f(s)e^{-\gamma(s)}(N_1^{u_1} - \varepsilon)u_2^{(1)}(t - s, x) \, ds - r_2u_2^{(1)}(t, x) - a_{22}(u_2^{(1)}(t, x))^2.
\]

By comparison we have \(u_2^{(1)}(t, x) \geq v_2^{(1)}(t, x), t > t_3 + \tau, x \in \Omega\), where \(v_2^{(1)}(t, x)\) is the solution of the following problem:
\[
\begin{align*}
\frac{\partial v_2^{(1)}}{\partial t} &= \Delta v_2^{(1)} + \alpha N_1^{u_1} - \varepsilon \int_0^t f(s)e^{-\gamma(s)}v_2^{(1)}(t - s, x) \, ds \\
&\quad - r_2v_2^{(1)}(t, x) - a_{22}(v_2^{(1)}(t, x))^2, \quad t > t_3 + \tau, \ x \in \Omega, \\
\frac{\partial v_2^{(1)}}{\partial v} &= 0, \quad t > t_3 + \tau, \ x \in \partial \Omega, \\
v_2^{(1)}(t, x) &= \frac{1}{2} u_2(t, x), \quad t \in [-\tau, t_3 + \tau], \ x \in \bar{\Omega}.
\end{align*}
\]

By Lemma 2.2 it follows that
\[
\lim_{t \to +\infty} v_2^{(1)}(t, x) = \frac{\alpha(N_1^{u_1} - \varepsilon)\int_0^t f(s)e^{-\gamma(s)} \, ds - r_2}{a_{22}} \text{ uniformly for } x \in \bar{\Omega}.
\]
Since it is true for arbitrary $\varepsilon > 0$ sufficiently small, we have
\[
\liminf_{t \to +\infty} \min_{x \in \Omega} u_2(t, x) \geq \frac{2N_1^\varepsilon \int_0^T f(s)e^{-\gamma s} ds - r_2}{a_{22}} := N_1^\varepsilon,
\]
(2.17)
and for $\forall \varepsilon > 0$ sufficiently small, there is a $t_4 > t_3 + \tau$ such that if $t > t_4$,
\[
\min_{x \in \Omega} u_2^{(1)}(t, x) \geq N_1^\varepsilon - \varepsilon.
\]
(2.18)
Let $(\bar{u}_1^{(2)}(t, x), \bar{u}_2^{(2)}(t, x))$ be the solution of the following problem:
\[
\begin{align*}
\frac{\partial \bar{u}_1^{(2)}}{\partial t} &= \Delta \bar{u}_1^{(2)} + \bar{u}_1^{(2)}(t, x)[r_1 - a_{11}\bar{u}_1^{(2)}(t, x) - a_{12}\bar{u}_2^{(1)}(t, x)], & t > t_4, & x \in \Omega, \\
\frac{\partial \bar{u}_2^{(2)}}{\partial t} &= \Delta \bar{u}_2^{(2)} + \gamma \int_0^T f(s)e^{-\gamma s} \bar{u}_1^{(2)}(t - s, x)\bar{u}_2^{(2)}(t - s, x) ds \\
&- r_2 \bar{u}_2^{(2)}(t, x) - a_{22}(\bar{u}_2^{(2)}(t, x))^2, & t > t_4, & x \in \Omega,
\end{align*}
\]
(2.19)
Thus, $(\bar{u}_1^{(1)}(t, x), \bar{u}_2^{(2)}(t, x))$ and $(\bar{u}_1^{(2)}(t, x), \bar{u}_2^{(2)}(t, x))$ are a pair of coupled upper and lower solutions of problem (1.4)–(1.6). By Lemma 2.1 we have
\[
\bar{u}_1^{(1)}(t, x) \leq u_1(t, x) \leq \bar{u}_1^{(2)}(t, x), \quad \bar{u}_2^{(1)}(t, x) \leq u_2(t, x) \leq \bar{u}_2^{(2)}(t, x).
\]
For $\forall \varepsilon > 0$ sufficiently small, it follows from (2.18) and the first equation of system (2.19) that for $t > t_4$, $x \in \Omega$,
\[
\frac{\partial \bar{u}_1^{(2)}}{\partial t} \leq \Delta \bar{u}_1^{(2)} + \bar{u}_1^{(2)}(t, x)[r_1 - a_{11}\bar{u}_1^{(2)}(t, x) - a_{12}[N_1^\varepsilon - \varepsilon]].
\]
By comparison we get $\bar{u}_1^{(2)}(t, x) \leq \omega_1^{(2)}(t, x), t > t_4, x \in \Omega$, where $\omega_1^{(2)}(t, x)$ is the solution of the following problem:
\[
\begin{align*}
\frac{\partial \omega_1^{(2)}}{\partial t} &= \Delta \omega_1^{(2)} + \epsilon \omega_1^{(2)}(t, x)[r_1 - a_{11}\omega_1^{(2)}(t, x) - a_{12}[N_1^\varepsilon - \varepsilon]], & t > t_4, & x \in \Omega, \\
\frac{\partial \omega_1^{(2)}}{\partial v} &= 0, & t > t_4, & x \in \partial \Omega, \quad \omega_1^{(2)}(t, x) = K_1, \quad (t, x) \in [-\tau, t_4] \times \bar{\Omega}.
\end{align*}
\]
(2.20)
We derive from (2.20) that
\[
\lim_{t \to +\infty} \omega_1^{(2)}(t, x) = \frac{r_1 - a_{12}[N_1^\varepsilon - \varepsilon]}{a_{11}} \quad \text{uniformly for } x \in \bar{\Omega}.
\]
Since it is true for arbitrary $\varepsilon > 0$ sufficiently small, we can conclude that
\[
\bar{U}_1 = \limsup_{t \to +\infty} \max_{x \in \Omega} u_1(t, x) \leq \frac{r_1 - a_{12}[N_1^\varepsilon]}{a_{11}} := M_2^\varepsilon
\]
(2.21)
and for $\forall \varepsilon > 0$ sufficiently small, there is a $t_5 > t_4$ such that if $t > t_5$,
\[
\max_{x \in \Omega} u_2^{(1)}(t, x) \leq M_2^\varepsilon + \varepsilon.
\]
(2.22)
For $\forall \varepsilon > 0$ sufficiently small, it follows from (2.22) and the second equation of (2.19) that for $t > t_5 + \tau$, $x \in \Omega$,
\[
\frac{\partial \bar{u}_2^{(2)}}{\partial t} \leq \Delta \bar{u}_2^{(2)} + \gamma \int_0^T f(s)e^{-\gamma s}(M_2^\varepsilon + \varepsilon)\bar{u}_2^{(2)}(t - s, x) ds - r_2 \bar{u}_2^{(2)}(t, x) - a_{22}(\bar{u}_2^{(2)}(t, x))^2.
\]
We consider the following auxiliary problem:
\[
\frac{\partial \omega_2^{(2)}}{\partial t} = \Delta \omega_2^{(2)} + x(M_2^{u_1} + \varepsilon) \int_0^t f(s) e^{-\tau s} \omega_2^{(2)}(t - s, x) \, ds \\
- r_2 \omega_2^{(2)}(t, x) - a_{22}(\omega_2^{(2)}(t, x))^2, \quad t > t_5 + \tau, \ x \in \Omega,
\]
(2.23)
\[
\frac{\partial \omega_2^{(2)}}{\partial v} = 0, \quad t > t_5 + \tau, \ x \in \partial \Omega,
\]
\[
\omega_2^{(2)}(t, x) = K_2, \quad t \in [-\tau, t_5 + \tau], \ x \in \bar{\Omega}.
\]
By Lemma 2.2 it follows from (2.23) that
\[
\lim_{t \to +\infty} \omega_2^{(2)}(t, x) = \frac{x(M_2^{u_1} + \varepsilon) \int_0^t f(s) e^{-\tau s} ds - r_2}{a_{22}} \text{ uniformly for } x \in \Omega.
\]
Since it is true for arbitrary \( \varepsilon > 0 \) sufficiently small, we conclude that
\[
\bar{U}_2 = \limsup_{t \to +\infty} \max_{x \in \Omega} u_2^{(2)}(t, x) \leq \frac{x(M_2^{u_1} + \varepsilon) \int_0^t f(s) e^{-\tau s} ds - r_2}{a_{22}} := M_2^{u_2},
\]
(2.24)
and for \( \forall \varepsilon > 0 \) sufficiently small, there is a \( t_6 > t_5 + \tau \) such that
\[
\max_{x \in \Omega} u_2^{(2)}(t, x) \leq M_2^{u_2} + \varepsilon.
\]
(2.25)
Let \((u_1^{(2)}(t, x), u_2^{(2)}(t, x))\) be the solution of the following problem:
\[
\frac{\partial u_1^{(2)}}{\partial t} = \Delta u_1^{(2)} + u_1^{(2)}(t, x)[r_1 - a_{11}u_1^{(2)}(t, x) - a_{12}u_2^{(2)}(t, x)], \quad t > t_6, \ x \in \Omega,
\]
\[
\frac{\partial u_1^{(2)}}{\partial v} = 0, \quad t > t_6, \ x \in \partial \Omega,
\]
\[
u_1^{(2)}(t, x) = \frac{1}{2} u_1(t, x), \quad \nu_2^{(2)}(t, x) = \frac{1}{2} u_2(t, x), \quad t \in [-\tau, t_6], \ x \in \bar{\Omega}.
\]
(2.26)
It is easy to see that \((u_1^{(2)}(t, x), u_2^{(2)}(t, x))\) and \((v_1^{(2)}(t, x), v_2^{(2)}(t, x))\) are a pair of coupled upper and lower solutions of problem (1.4)–(1.6). By Lemma 2.1 we derive
\[
u_1^{(2)}(t, x) \leq u_1(t, x) \leq \bar{u}_1^{(2)}(t, x), \quad v_2^{(2)}(t, x) \leq u_2(t, x) \leq \bar{u}_2^{(2)}(t, x).
\]
It follows from (2.25) and the first equation of problem (2.26) that for \( t > t_6, x \in \Omega \),
\[
\frac{\partial u_1^{(2)}}{\partial t} \geq \Delta u_1^{(2)} + u_1^{(2)}(t, x)[r_1 - a_{11}u_1^{(2)}(t, x) - a_{12}(M_2^{u_2} + \varepsilon)].
\]
By comparison we get \( u_1^{(2)}(t, x) \geq v_1^{(2)}(t, x), t > t_6, x \in \Omega \), where \( v_1^{(2)}(t, x) \) is the solution of the following problem:
\[
\frac{\partial v_1^{(2)}}{\partial t} = \Delta v_1^{(2)} + v_1^{(2)}(t, x)[r_1 - a_{11}v_1^{(2)}(t, x) - a_{12}(M_2^{u_2} + \varepsilon)], \quad t > t_6, \ x \in \Omega,
\]
\[
\frac{\partial v_1^{(2)}}{\partial v} = 0, \quad t > t_6, \ x \in \partial \Omega,
\]
\[
v_1^{(2)}(t, x) = \frac{1}{2} u_1(t, x), \quad (t, x) \in [-\tau, t_6] \times \bar{\Omega}.
\]
(2.27)
We derive from (2.27) that
\[
\lim_{t \to +\infty} v_{1}^{(2)}(t, x) = \frac{r_{1} - a_{12}(M_{2}^{u} + \varepsilon)}{a_{11}} \quad \text{uniformly for } x \in \Omega.
\]
Since this is true for arbitrary \(\varepsilon > 0\) sufficiently small, we therefore conclude that
\[
U_{1} = \liminf_{t \to +\infty} \min_{x \in \Omega} u_{1}(t, x) \geq \frac{r_{1} - a_{12}M_{2}^{u}}{a_{11}} := N_{2}^{u},
\]
and for \(\forall \varepsilon > 0\) sufficiently small, there is a \(t_{7} > t_{6}\) such that if \(t > t_{7}\),
\[
\min_{x \in \Omega} u_{1}^{(2)}(t, x) \geq N_{2}^{u} - \varepsilon.
\]
Thus, for \(\forall \varepsilon > 0\) sufficiently small, it follows from (2.29) and the second equation of problem (2.26) that for \(t > t_{7} + \tau, x \in \Omega\),
\[
\frac{\partial u_{2}^{(2)}}{\partial t} = \Delta u_{2}^{(2)} + \alpha \int_{0}^{\tau} f(s)e^{-\gamma(s)}(N_{2}^{u} - \varepsilon)u_{2}^{(2)}(t - s, x)ds - r_{22}u_{2}^{(2)}(t, x) - a_{22}u_{2}^{(2)}(t, x)^{2}.
\]
By comparison we get \(u_{2}^{(2)}(t, x) \geq v_{2}^{(2)}(t, x), t > t_{7} + \tau, x \in \Omega\), where \(v_{2}^{(2)}(t, x)\) is the solution of the following problem:
\[
\begin{align*}
\frac{\partial v_{2}^{(2)}}{\partial t} &= \Delta v_{2}^{(2)} + \alpha(N_{2}^{u} - \varepsilon)\int_{0}^{\tau} f(s)e^{-\gamma(s)}v_{2}^{(2)}(t - s, x)ds \\
&\quad - r_{22}v_{2}^{(2)}(t, x) - a_{22}v_{2}^{(2)}(t, x)^{2}, \quad t > t_{7} + \tau, \ x \in \Omega, \\
\frac{\partial v_{2}^{(2)}}{\partial \nu} &= 0, \quad t > t_{7} + \tau, \ x \in \partial \Omega, \\
v_{2}^{(2)}(t, x) &= \frac{1}{2}u_{2}(t, x), \quad t \in [-\tau, t_{7} + \tau], \ x \in \bar{\Omega}.
\end{align*}
\]
By Lemma 2.2 it follows that
\[
\lim_{t \to +\infty} v_{2}^{(2)}(t, x) = \frac{\alpha(N_{2}^{u} - \varepsilon)\int_{0}^{\tau} f(s)e^{-\gamma(s)}ds - r_{2}}{a_{22}} \quad \text{uniformly for } x \in \Omega.
\]
Since it is true for arbitrary \(\varepsilon > 0\) sufficiently small, we have
\[
\liminf_{t \to +\infty} \min_{x \in \Omega} u_{2}(t, x) \geq \frac{\alpha(N_{2}^{u} - \varepsilon)\int_{0}^{\tau} f(s)e^{-\gamma(s)}ds - r_{2}}{a_{22}} := N_{2}^{u}.
\]
Continuing this process, we get four sequences \(M_{n}^{u_{1}}, M_{n}^{u_{2}}, N_{n}^{u_{1}}, N_{n}^{u_{2}}\) \((n = 1, 2, \ldots)\), such that for \(n \geq 2\),
\[
\begin{align*}
M_{n}^{u_{1}} &= \frac{r_{1} - a_{12}N_{n-1}^{u_{2}}}{a_{11}}, \\
M_{n}^{u_{2}} &= \frac{\alpha\int_{0}^{\tau} f(s)e^{-\gamma(s)}dsM_{n}^{u_{1}} - r_{2}}{a_{22}}, \\
N_{n}^{u_{1}} &= \frac{r_{1} - a_{12}M_{n}^{u_{2}}}{a_{11}}, \\
N_{n}^{u_{2}} &= \frac{\alpha\int_{0}^{\tau} f(s)e^{-\gamma(s)}dsN_{n}^{u_{1}} - r_{2}}{a_{22}}.
\end{align*}
\]
It is readily seen that
\[
N_{n}^{u_{i}} \leq \liminf_{t \to +\infty} \min_{x \in \Omega} u_{i}(t, x) \leq \limsup_{t \to +\infty} \max_{x \in \Omega} u_{i}(t, x) \leq M_{n}^{u_{i}} \quad (i = 1, 2).
\]
We derive from (2.32) that

\[ M_{n+1}^u = \frac{(a_1 a_{22} - a_{12} \int_0^t f(s)e^{-\tau s} \, ds)(r_1 a_{22} + r_2 a_{12})}{a_1^2 a_{22}^2} + \frac{a_{12}^2 (\int_0^t f(s)e^{-\tau s} \, ds)^2}{a_1^2 a_{22}^2} M_n^u. \]  

(2.34)

Noting that \( M_{n+1}^u \geq u_1^* \) and \( a_{12} a_{22} > a_{12} \int_0^t f(s)e^{-\tau s} \, ds \), it follows from (2.34) that

\[ M_{n+1}^u - M_n^u = \left( \frac{(a_1 a_{22} - a_{12} \int_0^t f(s)e^{-\tau s} \, ds)^2}{a_1^2 a_{22}^2} u_1^* \right) - \left( \frac{a_{12}^2 (\int_0^t f(s)e^{-\tau s} \, ds)^2}{a_1^2 a_{22}^2} \right) u_1^* + \left[ \frac{a_{12}^2 (\int_0^t f(s)e^{-\tau s} \, ds)^2}{a_1^2 a_{22}^2} - 1 \right] M_n^u \leq 0. \]

Therefore, the sequence \( M_n^u \) is monotonically decreasing. Accordingly, \( \lim_{n \to +\infty} M_n^u \) exists. Taking \( n \to +\infty \), we derive from (2.34) that

\[ \lim_{n \to +\infty} M_n^u = \frac{r_1 a_{22} + r_2 a_{12}}{a_1 a_{22} + a_{12} \int_0^t f(s)e^{-\tau s} \, ds} = u_1^*. \]  

(2.35)

We therefore obtain from (2.32) and (2.35) that

\[ \lim_{n \to +\infty} M_n^u = u_2^*, \quad \lim_{n \to +\infty} N_n^u = u_1^*, \quad \lim_{n \to +\infty} N_n^u = u_2^*. \]

This completes the proof. \( \square \)

**Theorem 2.2.** Let the initial functions \( \phi_i (i = 1, 2) \) be Hölder continuous in \([-\tau, 0] \times \bar{\Omega} \), with \( \phi_i(t, x) \geq 0, \phi_i(0, x) \neq 0 \). Let \( (u_1(t, x), u_2(t, x)) \) satisfy (1.4)–(1.6). In addition to (H1), assume further that

\( \text{(H4)} \) \( r_1 x \int_0^t f(s)e^{-\tau s} \, ds < r_2 a_{11}. \)

Then

\[ \lim_{t \to +\infty} u_1(t, x) = \frac{r_1}{a_{11}}, \quad \lim_{t \to +\infty} u_2(t, x) = 0 \]

uniformly for \( x \in \bar{\Omega} \).

**Proof.** Let (H4) hold. Choose \( \varepsilon > 0 \) sufficiently small such that the following holds:

\[ \alpha \left( \frac{r_1}{a_{11}} + \varepsilon \right) \int_0^t f(s)e^{-\tau s} \, ds < r_2. \]  

(2.36)

Let \( (\bar{u}_1^{(1)}(t, x), \bar{u}_2^{(1)}(t, x)) \) be the solution of the following problem:

\[ \frac{\partial \bar{u}_1^{(1)}}{\partial t} = \Delta \bar{u}_1^{(1)} + \bar{u}_1^{(1)}(t, x)[r_1 - a_{11} \bar{u}_1^{(1)}(t, x)], \quad t > 0, \quad x \in \Omega, \]

\[ \frac{\partial \bar{u}_2^{(1)}}{\partial t} = \Delta \bar{u}_2^{(1)} + \alpha \int_0^t f(s)e^{-\tau s} \bar{u}_1^{(1)}(t, x) \bar{u}_2^{(1)}(t, x) \, ds \]

\[ - r_2 \bar{u}_2^{(1)}(t, x) - a_{22} \bar{u}_2^{(1)}(t, x)^2, \quad t > 0, \quad x \in \Omega, \]

\[ \frac{\partial \bar{u}_1^{(1)}}{\partial v} = \frac{\partial \bar{u}_2^{(1)}}{\partial v} = 0, \quad t > 0, \quad x \in \partial \Omega, \]

\[ \bar{u}_1^{(1)}(t, x) = K_1, \quad \bar{u}_2^{(1)}(t, x) = K_2, \quad t \in [-\tau, 0], \quad x \in \bar{\Omega}. \]

Then \( (0, 0) \) and \( (\bar{u}_1^{(1)}(t, x), \bar{u}_2^{(1)}(t, x)) \) are a pair of coupled lower and upper solutions of problem (1.4)–(1.6). Therefore, by Lemma 2.1 we derive that

\[ 0 \leq u_1(t, x) \leq \bar{u}_1^{(1)}(t, x), \quad 0 \leq u_2(t, x) \leq \bar{u}_2^{(1)}(t, x). \]
It follows from the first equation of system (2.37) that
\[
\lim_{t \to +\infty} \hat{u}^{(1)}_1(t,x) = \frac{r_1}{a_{11}} \text{ uniformly for } x \in \Omega.
\]

Hence, for \( \forall \varepsilon > 0 \) sufficiently small satisfying (2.36), there exists a \( T_1 > 0 \) such that if \( t > T_1 \)
\[
\max_{x \in \Omega} \hat{u}^{(1)}_1(t,x) < \frac{r_1}{a_{11}} + \varepsilon. \tag{2.38}
\]

Since this is true for arbitrary \( \varepsilon > 0 \) sufficiently small, we conclude that
\[
\limsup_{t \to +\infty} \max_{x \in \Omega} u_1(t,x) \leq \frac{r_1}{a_{11}}. \tag{2.39}
\]

We consider the following auxiliary problem:
\[
\frac{\partial \omega^{(1)}_2}{\partial t} = \Delta \omega^{(1)}_2 + \alpha \left( \frac{r_1}{a_{11}} + \varepsilon \right) \int_0^t f(s) e^{-\gamma s} \omega^{(1)}_2(t-s,x) \, ds
- r_2 \omega^{(1)}_2(t,x) - a_{22} (\omega^{(1)}_2(t,x))^2, \quad t > T_1, \ x \in \Omega,
\]
\[
\frac{\partial \omega^{(1)}_2}{\partial v} = 0, \quad t > T_1, \ x \in \partial \Omega,
\omega^{(1)}_2(t,x) = K_2, \quad t \in [-\tau, T_1], \ x \in \Omega. \tag{2.40}
\]

By Lemma 2.2, it follows from (2.36) and (2.40) that
\[
\lim_{t \to +\infty} \omega^{(1)}_2(t,x) = 0 \text{ uniformly for } x \in \Omega.
\]

We therefore have
\[
\lim_{t \to +\infty} \hat{u}^{(1)}_2(t,x) = 0 \text{ uniformly for } x \in \Omega.
\]

Therefore, for \( \forall \varepsilon > 0 \) sufficiently small, there is a \( T_2 > T_1 > 0 \) such that if \( t > T_2 \),
\[
0 < \max_{x \in \Omega} \hat{u}^{(1)}_2(t,x) < \varepsilon, \tag{2.41}
\]

and
\[
\lim_{t \to +\infty} u_2(t,x) = 0 \text{ uniformly for } x \in \Omega.
\]

Define \((\hat{u}^{(1)}_1(t,x), \hat{u}^{(1)}_2(t,x))\) by
\[
\frac{\partial \hat{u}^{(1)}_1}{\partial t} = \Delta \hat{u}^{(1)}_1 + \hat{u}^{(1)}_1(t,x)[r_1 - a_{11} \hat{u}^{(1)}_1(t,x) - a_{12} \hat{u}^{(1)}_2(t,x)], \quad t > T_2, \ x \in \Omega,
\]
\[
\frac{\partial \hat{u}^{(1)}_2}{\partial t} = \Delta \hat{u}^{(1)}_2 + \alpha \int_0^t f(s) e^{-\gamma s} \hat{u}^{(1)}_2(t-s,x) \hat{u}^{(1)}_2(t-s,x) \, ds
- r_2 \hat{u}^{(1)}_2(t,x) - a_{22} (\hat{u}^{(1)}_2(t,x))^2, \quad t > T_2, \ x \in \Omega,
\]
\[
\frac{\partial \hat{u}^{(1)}_1}{\partial v} = \frac{\partial \hat{u}^{(1)}_2}{\partial v} = 0, \quad t > T_2, \ x \in \partial \Omega,
\hat{u}^{(1)}_1(t,x) = \frac{1}{2} u_1(t,x), \quad \hat{u}^{(1)}_2(t,x) = \frac{1}{2} u_2(t,x), \quad t \in [-\tau, T_2], \ x \in \Omega. \tag{2.42}
\]

Then \((\hat{u}^{(1)}_1, \hat{u}^{(1)}_2)\) and \((\hat{u}^{(1)}_1, \hat{u}^{(1)}_2)\) are a pair of coupled lower and upper solutions of problem (1.4)–(1.6). By Lemma 2.1 it follows that
\[
u^{(1)}_1(t,x) \leq u_1(t,x) \leq \hat{u}^{(1)}_1(t,x), \quad \hat{u}^{(1)}_2(t,x) \leq u_2(t,x) \leq \hat{u}^{(1)}_2(t,x).
\]
For $\forall \varepsilon > 0$ sufficiently small, we derive from (2.41) and the first equation of system (2.42) that for $t > T_2$, $x \in \Omega$,

$$\frac{\partial u_1^{(1)}}{\partial t} \geq \Delta u_1^{(1)} + u_1^{(1)}(t,x)[r_1 - a_{11}u_1^{(1)}(t,x) - a_{12}\varepsilon].$$

By comparison we get $u_1^{(1)}(t,x) \geq v_1^{(1)}(t,x), t > T_2, x \in \Omega$, where $v_1^{(1)}(t,x)$ is the solution of the following problem:

$$\frac{\partial v_1^{(1)}}{\partial t} = \Delta v_1^{(1)} + v_1^{(1)}(t,x)[r_1 - a_{11}v_1^{(1)}(t,x) - a_{12}\varepsilon], \quad t > T_2, \ x \in \Omega,$n

$$\frac{\partial v_1^{(1)}}{\partial \nu} = 0, \quad t > T_2, \ x \in \partial \Omega,$n

$$v_1^{(1)}(t,x) = \frac{1}{2}u_1(t,x), \quad (t,x) \in [-\tau, T_2] \times \Omega.$$

It is easy to see that

$$\lim_{t \to +\infty} v_1^{(1)}(t,x) = \frac{r_1 - a_{12}\varepsilon}{a_{11}} \text{ uniformly for } x \in \Omega.$$

Since this is true for arbitrary $\varepsilon > 0$ sufficiently small, we therefore derive that

$$\liminf_{t \to +\infty} \min_{x \in \bar{\Omega}} u_1(t,x) \geq \frac{r_1}{a_{11}}. \quad (2.43)$$

It therefore follows from (2.39) and (2.43) that

$$\lim_{t \to +\infty} u_1(t,x) = \frac{r_1}{a_{11}} \text{ uniformly for } x \in \bar{\Omega}.$$

This completes the proof.  \( \square \)

3. Discussion

Motivated by the work of Al-Omari and Gourley [3] and Pao [12–14], in this paper, we incorporated spatial diffusion and stage structure for the predator into a Lotka–Volterra type predator–prey model. By using the coupled upper–lower solution technique, we derived sets of sufficient conditions to respectively guarantee that positive solutions of problem (1.4)–(1.6) either converge to the unique, positive, uniform equilibrium or to the semi-trivial uniform equilibrium. By Theorem 2.1, we see that if (H1)–(H3) hold, then positive solution of problem (1.4)–(1.6) converges to the unique, positive, uniform equilibrium $E^*$. Biologically, this implies that if: (i) the intrinsic growth rate of the prey and the conversion rate of the mature predator are high; (ii) the death rate of the mature predator and the intra-specific competition rate of the prey are sufficiently low; and (iii) the intra-specific competitions of both the prey and the mature predator dominate the inter-specific interaction between the prey and the mature predator, then the prey and the predator populations will coexist in a stable manner. By Theorem 2.2, we see that if (H3) holds, then positive solutions of problem (1.4)–(1.6) will converge to the semi-trivial uniform equilibriam of system (1.4). Ecologically, this implies that the predator population will go to extinction but the prey population will persist and this occurs if the death rate of the mature predator population and the intra-specific competition rate are high and the conversion rate of the predator and the intrinsic growth rate of the prey are sufficiently low. We would like to mention here that if (H2) holds but (H3) does not hold, then we anticipate that problem (1.4)–(1.6) will exhibit more complex dynamics, such as Hopf bifurcations. We did not pursue this further here and leave this for future consideration.
References