Analytic solution of the Black-Scholes equation.

This is a standard derivation which can be found in the textbooks.
The treatment here is similar to that in Wilmott, Howison and Dewynne.

If we consider a European call option which has strike price K at time T then we know that it value at time T is

$$C(S,T) = \max(S - K, 0)$$

We want to find its value at times $t<T$ so we need to solve the Black-Scholes equation

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0$$

with the above as boundary condition.

Note that if we drop the last two terms, we get a diffusion equation, but with a negative diffusion coefficient. However, the boundary condition is imposed at a future time, so this is a well-posed problem.

Since the stochastic process underlying the derivation of this equation is geometric Brownian motion, and we have seen that for this type of motion log*S* follows a Wiener process with constant coeffficients, then we might guess that the change of variable

$$S = Ke^{x}$$

would be useful. Scaling *S* in terms of *K* is not such an essential feature, but it simplifies the boundary condition.

We also make the problem into a forward diffusion problem by putting

$$\tau = \frac{1}{2}\sigma^{2}(T-t)$$

The boundary condition is now at $\tau=0$ and the problem is to be solved for $\tau>0$. The scaling factor is just introduced for convenience to get rid of some of the factors in the equation. We also scale the option price to K and write

$$C = Kv(x,t)$$

The result of this is the equation

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv$$

with

$$k = 2r/\sigma^2$$

and initial condition

$$v(x,0) = \max(e^x - 1, 0)$$

There is now just one dimensionless parameter *k* in the equation, so the various scalings of the variables produce a useful simplification.

The equation is now beginning to look like a diffusion equation. To reduce it to a diffusion equation we may note that if we put

$$v = e^{\alpha x + \beta\tau}$$

then

$$\frac{\partial v}{\partial \tau} = \beta u + \frac{\partial u}{\partial \tau}$$

$$\frac{\partial v}{\partial x} = \alpha u + \frac{\partial u}{\partial x}$$

$$\frac{\partial^2 v}{\partial x^2} = \alpha^2 u + 2\alpha \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}$$

We have two constants α and β to play with and we can choose them to make the terms in *u* and ∂*u*/∂*x* vanish. The required choices are

$$\alpha = -\frac{1}{2}(k-1)$$

$$\beta = -\frac{1}{4}(k+1)^2$$

The equation becomes a standard diffusion equation in *u* with *D*=1 and the boundary condition transforms to

$$u(x,0) = \max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0)$$

or

$$u(x,0) = e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x} \quad \text{if } x > 0, \text{ zero otherwise}$$

It is now a matter of writing down the solution we have already seen for the initial value problem for the diffusion equation in an unbounded domain then reversing all the transformations to get a solution in terms of the original variables of the problem.

The solution is

$$u(x,t) = \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} u(s,0) \exp\left(-\frac{(x-s)^2}{4\tau}\right) ds$$

It is convenient to make the change of variable

$$x' = (s-x)/\sqrt{2\tau}$$

which gives the following.

$$u(x,t) = I_1 - I_2$$

with

$$I_1 = \int_{-x/\sqrt{2\tau}}^{\infty} \exp\left[\frac{1}{2}(k+1)(x+x'\sqrt{2\tau})\right] e^{-\frac{1}{2}x'^2}\,dx'$$

$$I_2 = \int_{-x/\sqrt{2\tau}}^{\infty} \exp\left[\frac{1}{2}(k-1)(x+x'\sqrt{2\tau})\right] e^{-\frac{1}{2}x'^2}\,dx'$$

These integrals are evaluated by completing the square in the exponent. For the first of them we get

$$I_1 = \frac{1}{\sqrt{2\pi}} \exp\left[\frac{1}{2}(k+1) + \frac{1}{4}(k+1)^2 \tau \right]$$

$$\times \int_{-x\sqrt{2\tau}}^{\infty} \exp\{-\frac{1}{2}[x' - \frac{1}{2}(k+1)\sqrt{2\tau}\,]^2\}\,dx'$$

Now let

$$s = -(x' - \frac{1}{2}(k+1)\sqrt{2\tau}\,)$$

to make the integrand exp(-$s^2/2$).

This yields

$$I_1 = \exp[\frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2\tau]N(d_1)$$

where

$$d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau}$$

and

$$N(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1} e^{-\frac{1}{2}s^2}\,ds$$

is the cumulative distribution function for the normal distribution.

The other integral is the same with *k*-1 in place of *k*+1.

The final step is to go back though all the changes in variable to get the result in terms of our original parameters.

The result is

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

with

$$d_1 = \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

This is the Black-Scholes formula.

The corresponding formula for a European put option can be found similarly with the final condition

$$C(S,T) = \max(K - S, 0)$$

or, with less trouble, can be obtained from the put-call parity formula

$$C - P = S - Ke^{r(T-t)}$$

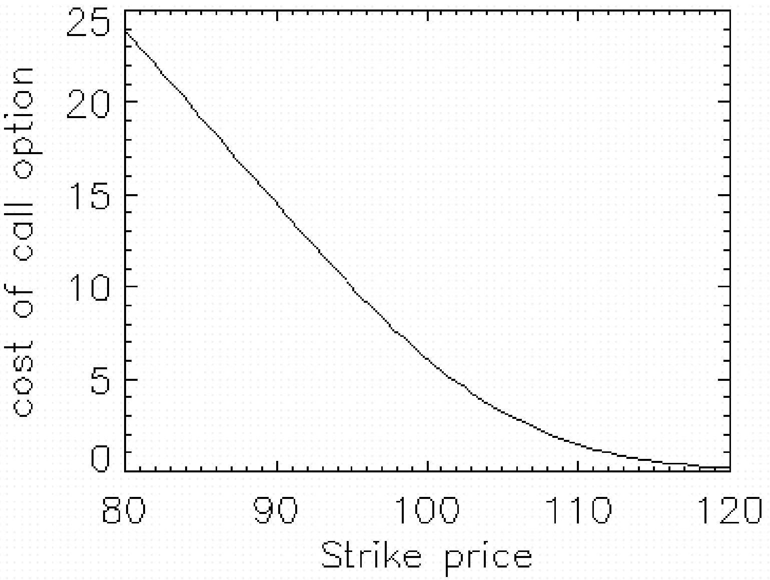
For more complicated options where the expiry time is not fixed or the payoff does not depend on a simple way on the value at some fixed time in the future numerical methods may be necessary.

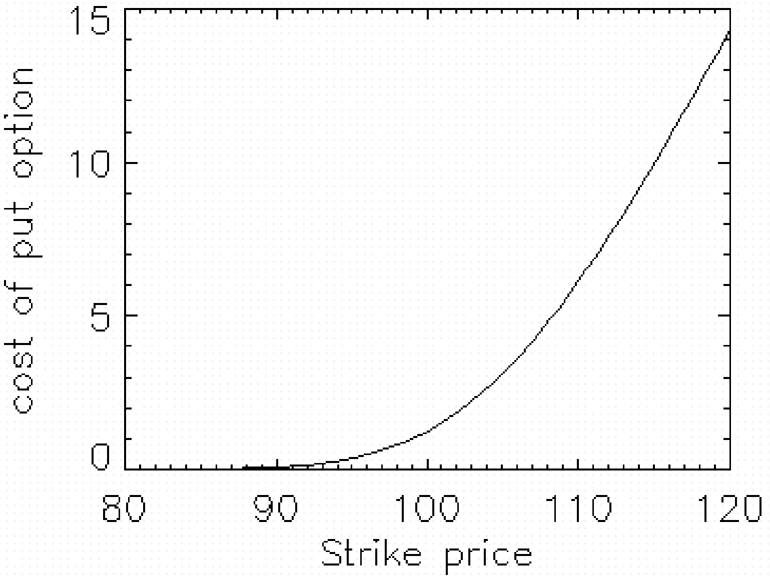
Most programming languages or packages like Maple have a built-in function or readily available routine for the cumulative normal distribution, so it is easy to write a program for evaluation of European options via the B-S formula.

The next slide shows a program written using IDL which uses the built in function called Gauss_pdf(x). It evaluates the cost of call and put options over any desired range of strike prices, prompting for input of the relevant parameters.

The page after that shows typical output with the current price 100, the interest rate 5%, the volatility 8% and the time to expiry one year.

```
pro BSFormula

; read in parameters

read,S,prompt='current price'
read,K1,K2,prompt='enter range of strike prices of interest'
read,r,prompt='annual interest rate %  '
read,sigma,prompt='volatility,% '
read,T,prompt='time to expiry in years'


;define arrays to hold results
K=make_array(51,/float)
C=make_array(51,/float)
P=make_array(51,/float)

;calculate cost as a function of strike price

r=r/100.

sigma=sigma/100.

for i=0,50 do begin
        K(i)=K1+i*(K2-k1)/50
        d1=(alog(S/K(i))+(r+0.5*sigma^2)*T)/(sigma*sqrt(T))
        d2=(alog(S/K(i))+(r-0.5*sigma^2)*T)/(sigma*sqrt(T))

        C(i)=S*Gauss_pdf(d1)-K(i)*exp(-r*T)*Gauss_pdf(d2)

        P(i)=C(i)-S+K(i)*exp(-r*T)

endfor

;plot results

        !p.multi=[0,1,2]

        window,xsize=475,ysize=700
        plot,K,C,color=0,background=16777215,xtitle='Strike price$
        ,ytitle='cost of call option',font=-1,charsize=2


        plot,K,P,color=0,background=16777215,xtitle='Strike price$
        ,ytitle='cost of put option',font=-1,charsize=2


end
```


cost of call option
25
20
15
10
5
0
80
90
100
110
120
Strike price


cost of put option
15
10
5
0
80
90
100
110
120
Strike price

The behaviour of these curves is easy to understand. If the strike price is far above the present price then it is unlikely that the price will rise enough for the call option to be exercised, so it has very little value. If the strike price is very low then there is a very good chance that the option will be exercised in which case it is valuable.

The straight line portion of the graph corresponds to the regime where S/K is large, making the *d's* large enough so that both the cumulative distribution functions in the formula take values close to 1.

Similar simple arguments apply to the put option curve.

Recall that in order to hedge risk, the person writing the option should hold a number Δ of shares, with

$$\Delta = \frac{\partial C}{\partial S}$$

so that it is important to know this quantity. Noting that

$$\frac{d}{dx}N(x) = \exp(-\frac{1}{2}x^2)$$

we can differentiate the B_S formula to get

$$\Delta = N(d_1) + S\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2}\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}e^{-\frac{1}{2}d_2^2}\frac{\partial d_2}{\partial S}$$

The last two terms here cancel as can be seen by noting that

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

that

$$\exp(-\frac{1}{2}d_1^2) = \exp(-\frac{1}{2}(d_1^2 - d_2^2))\exp(-\frac{1}{2}d_2^2)$$

and that

$$\exp[-\frac{1}{2}(d_1^2 - d_2^2)] = \exp[-\log(S/K) - r(T-t)]$$

$$= \frac{K}{S}\exp(-r(T-t))$$

Thus we get the simple formula

$$\Delta = N(d_1)$$

for a European call option and from the put-call parity relation we get

$$\Delta = N(d_1) - 1$$

for a European put option.